Stochastic Dynamic Matching in Graphs

Céline Comte

TU/e & LAAS–CNRS

SOLACE Seminar — January 26 and February 16, 2023

Outline

1. **Stochastic Matching: model, motivation, and notation**

2. **Performance under the first-come-first-matched policy**
   Comte, Stochastic Models (2022)

3. **Matching rates under an arbitrary policy**
   Comte, Mathieu, and Bušić, arXiv:2112.14457 (2022)

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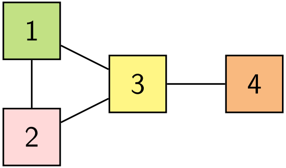
Compatibility graph

Graph $G = (V, E)$ undirected, connected, without self-loop



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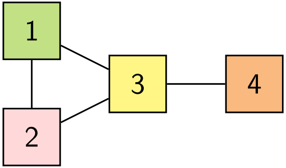

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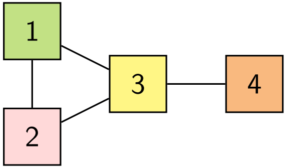

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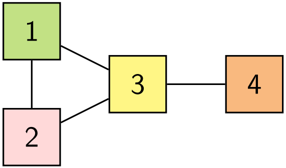

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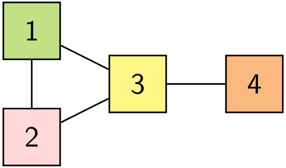

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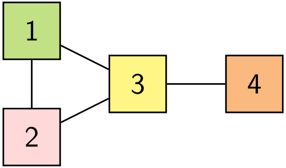

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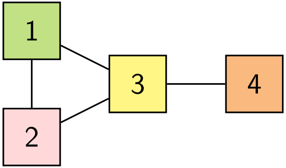

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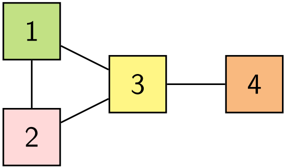
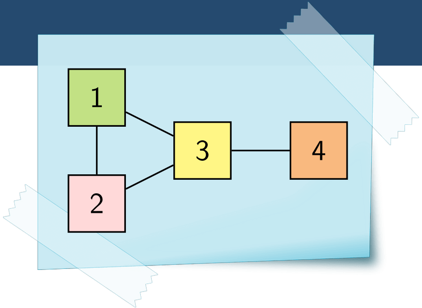


Random dynamics



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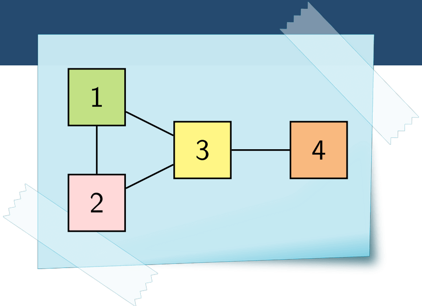

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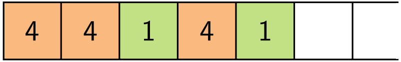

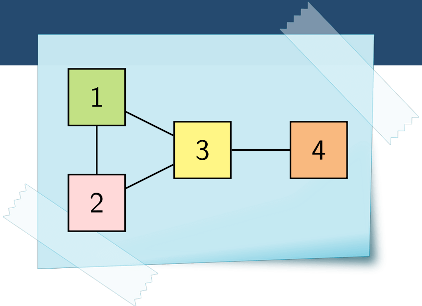

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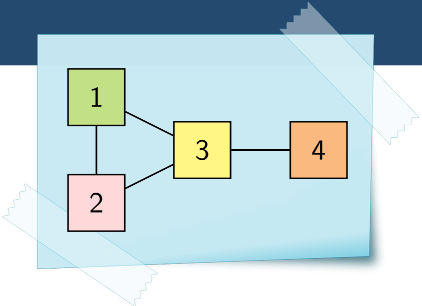

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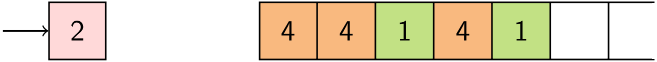

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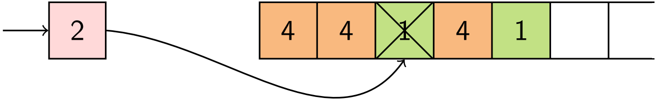

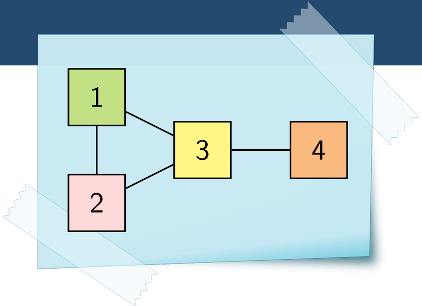

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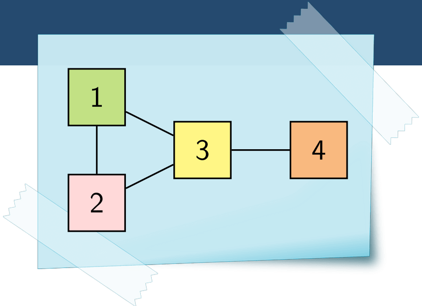

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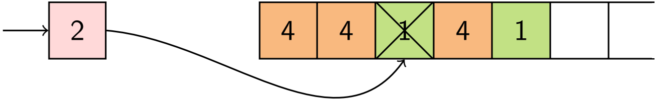

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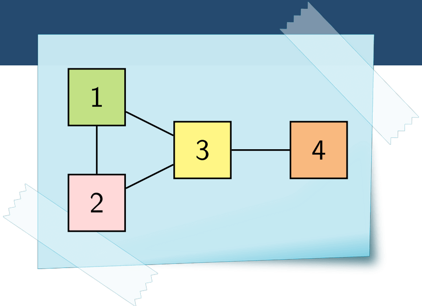

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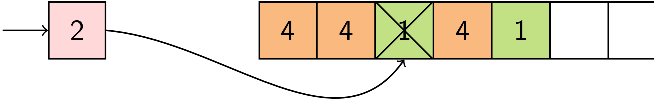

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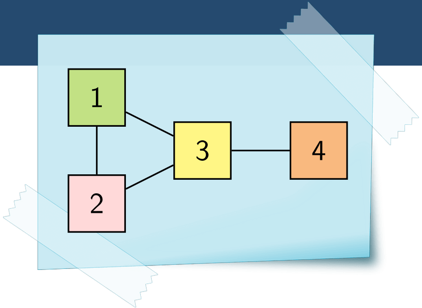
Notation:

- Arrival rate $\mu(U) = \sum_{i \in U} \mu_i$, $U \subseteq V$
- Load $\rho(I) = \frac{\mu(I)}{\mu(V(I))}$, $I \in \mathbb{I}$

“Stabilizability”



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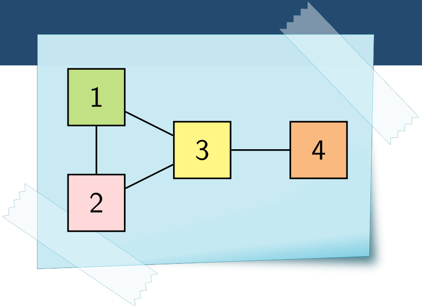

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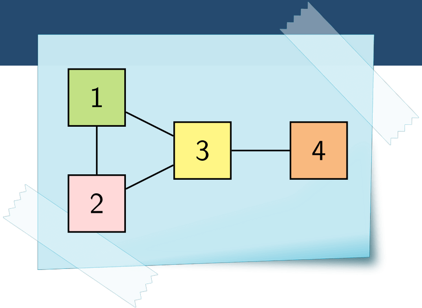

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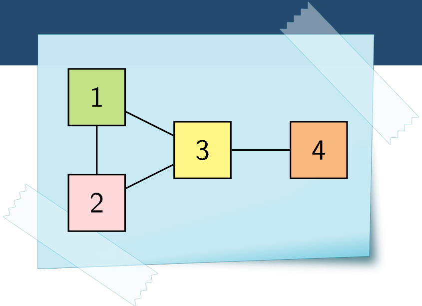

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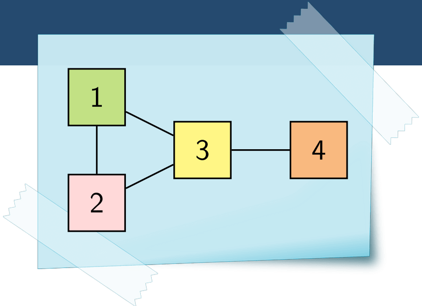

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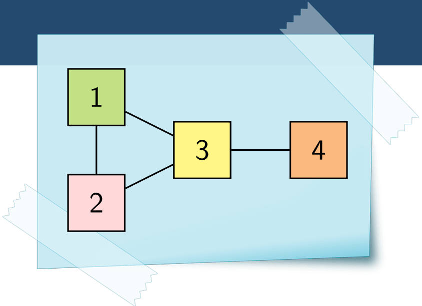

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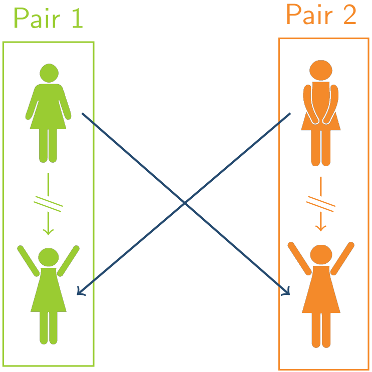
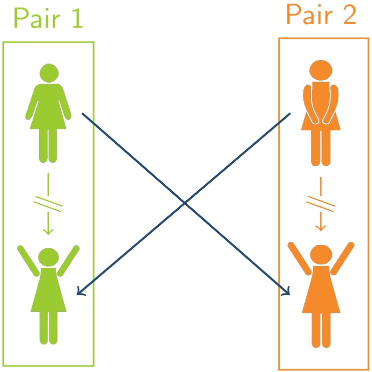
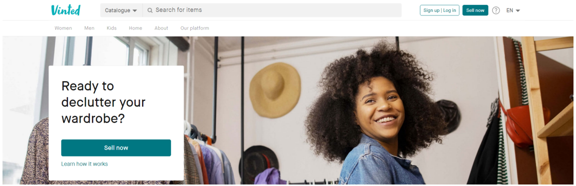


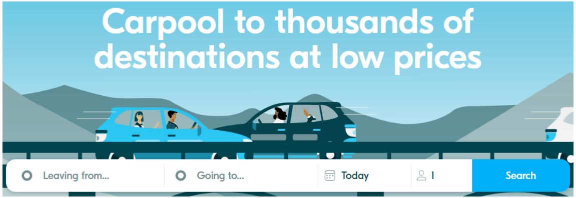
Applications

Paired kidney donation



Collaborative economy





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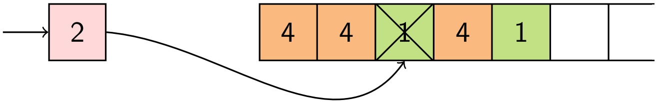
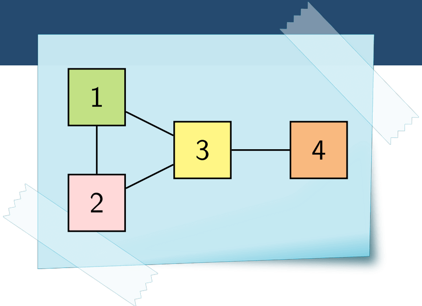
First-come-first-matched policy





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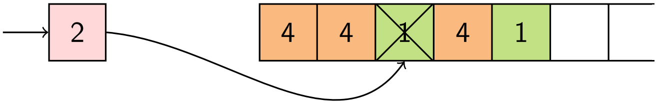

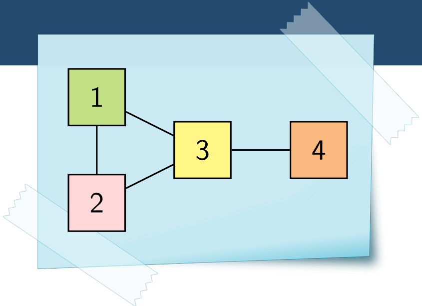

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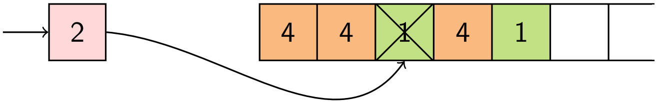

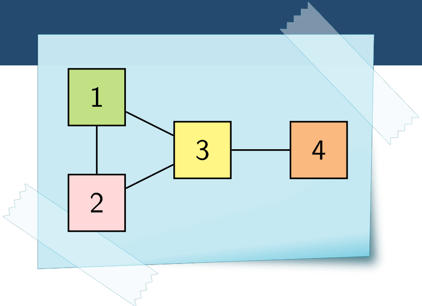

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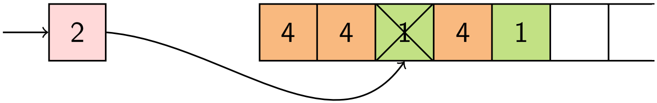

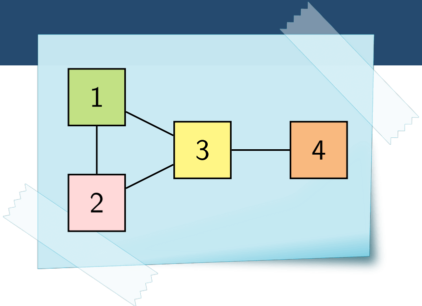

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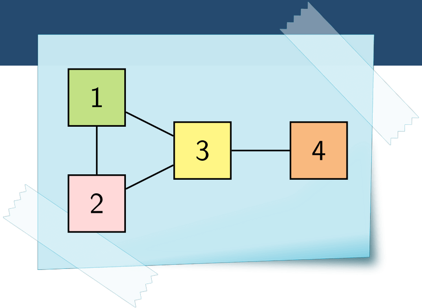
What is the long-term performance under first-come-first-matched?

Calculate long-term performance metrics



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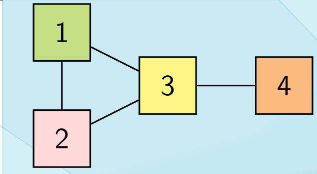

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- Stationary distribution of the set of unmatched classes:

$$\pi(I) = \frac{\rho(I)}{1-\rho(I)} \left(\sum_{i \in I} \frac{\mu_i}{\mu(I)} \pi(I \setminus \{i\}) \right), \quad I \in \mathbb{I}.$$

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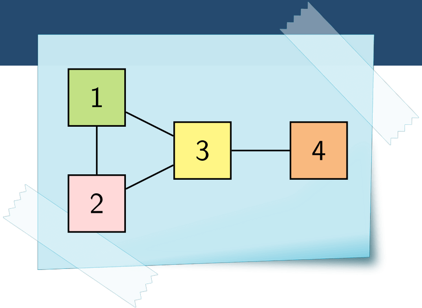

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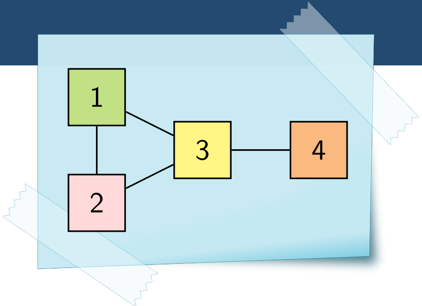

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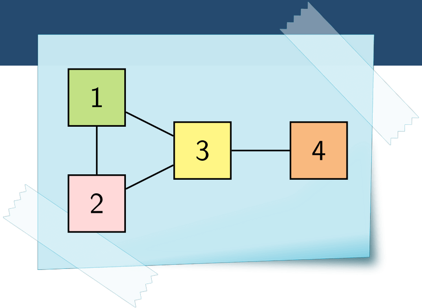

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  In particular, we obtain $\dfrac{\sum_{i \in V} \mu_i \omega_i}{\sum_{i \in V} \mu_i} = \dfrac{1}{2}.$

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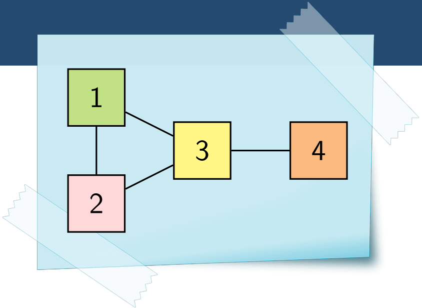

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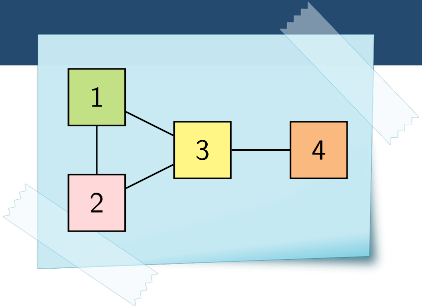

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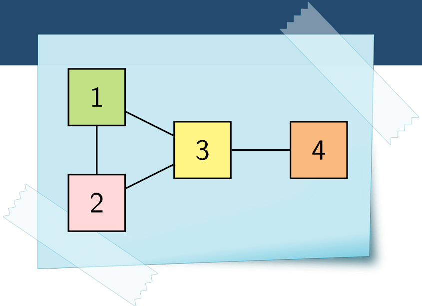
The mean waiting time of an item follows from Little's law.

Calculate long-term performance metrics



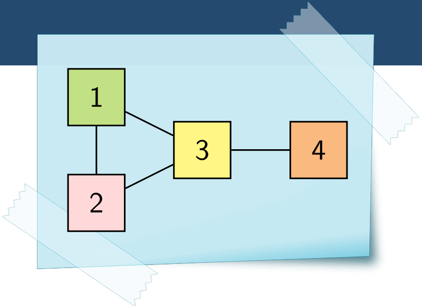
- Mean number of unmatched items:

$$L = \sum_{I \in \mathbb{I}} \ell(I),$$

with $$\ell(I) = \frac{\pi(I)}{1-\rho(I)} + \frac{\rho(I)}{1-\rho(I)} \left(\sum_{i \in I} \frac{\mu_i}{\mu(I)} \ell(I \setminus \{i\}) \right).$$

  The mean waiting time of an item follows from Little's law.
- More detailed formulas for the performance per class.

Calculate long-term performance metrics



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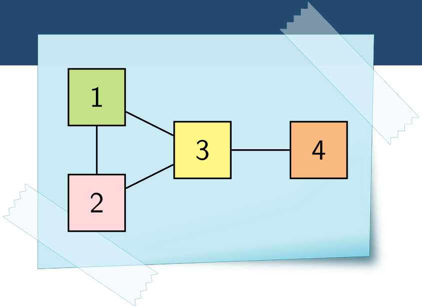
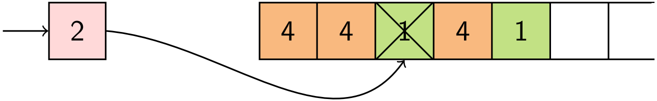
The mean waiting time of an item follows from Little's law.

- More detailed formulas for the performance per class.
- Similar results for stochastic bipartite matching model (Comte & Dorsman, ASMTA, 2021).

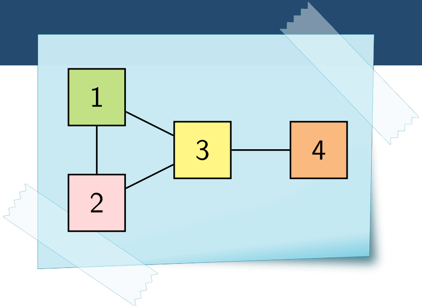
Calculate long-term performance metrics



- Matching rate along edge $k = \{i, j\}$: mean number of matches per time unit between classes i and j.



Calculate long-term performance metrics



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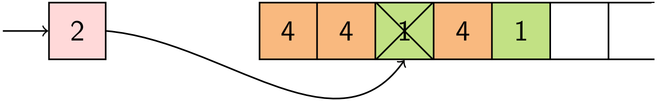
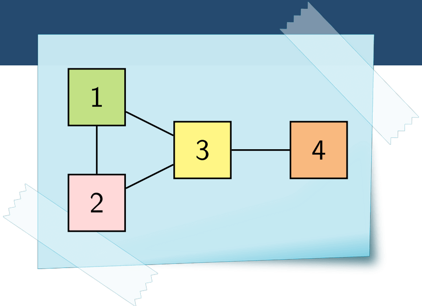


- Closed-form expression: consider a finer partition of the state space.

Calculate long-term performance metrics



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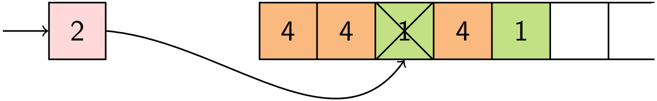
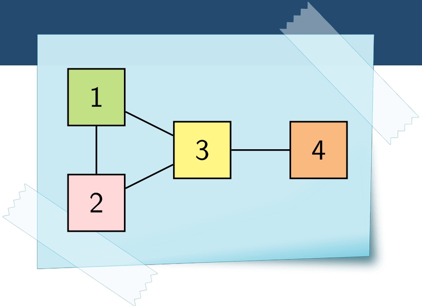


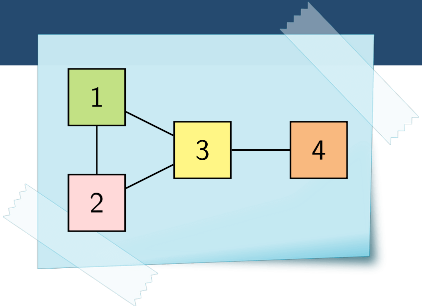
- Closed-form expression: consider a finer partition of the state space.
- Different approach in a few slides...

Heavy-traffic regime



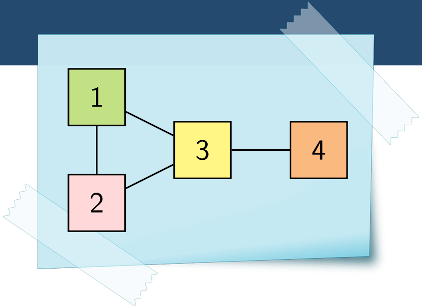
- Consider a **maximal** independent set $I \in \mathbb{I}$.

Heavy-traffic regime



- Consider a **maximal** independent set $I \in \mathbb{I}$.
- When the load $\rho(I) = \frac{\mu(I)}{\mu(V(I))}$ tends to 1,

Heavy-traffic regime



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  - the classes in I wait with probability 1,
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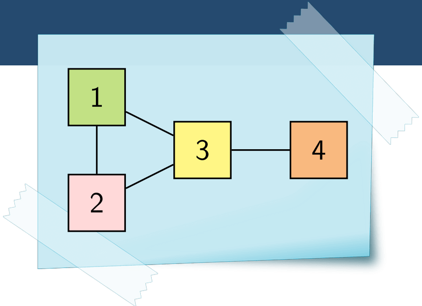
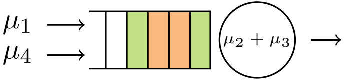
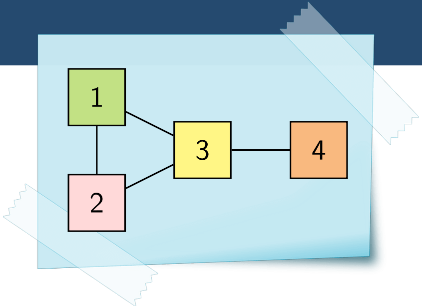


$\Downarrow$

M/M/1 multi-class queue



Heavy-traffic regime



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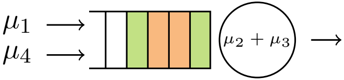

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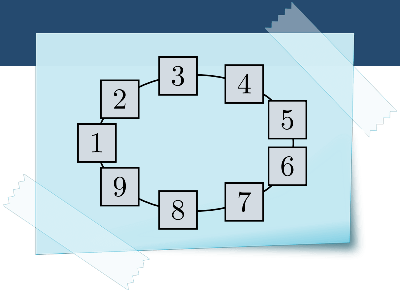
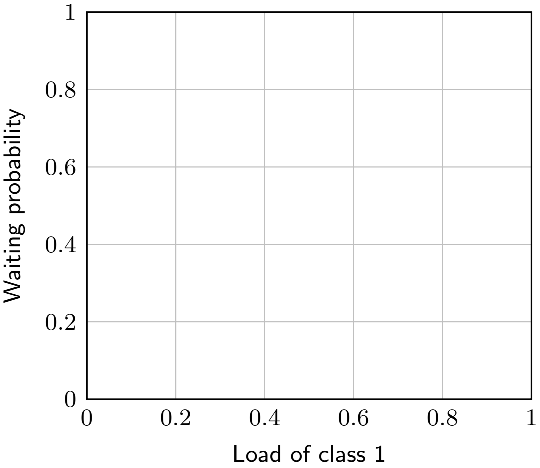
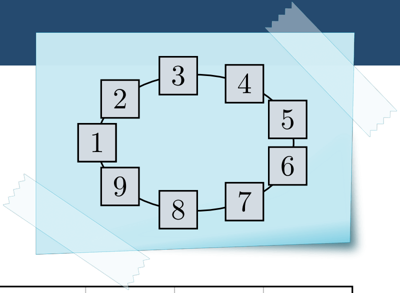
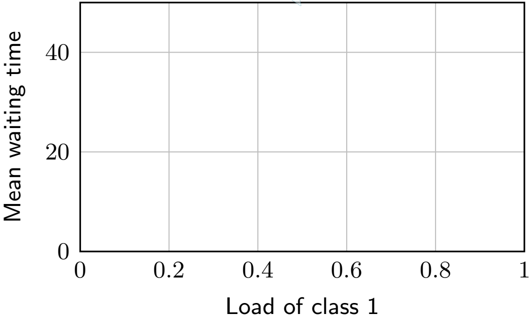
  - the classes in I wait with probability 1, while other classes wait with probability 0,
  - the mean number of unmatched items is $\sim \frac{\rho(I)}{1-\rho(I)}$.
- Take-away: minimizing the maximal load is a good heuristic to optimize performance.

Numerical results: Cycle

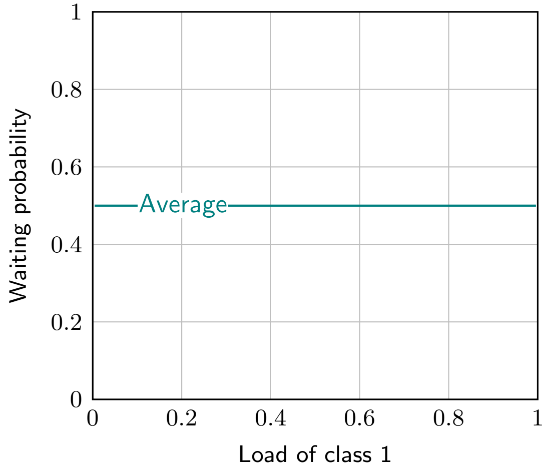


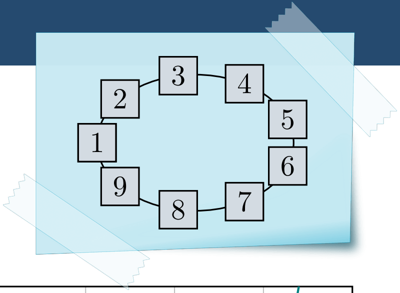

1
0.8
0.6
0.4
0.2
0
Waiting probability
0
0.2
0.4
0.6
0.8
1
Load of class 1

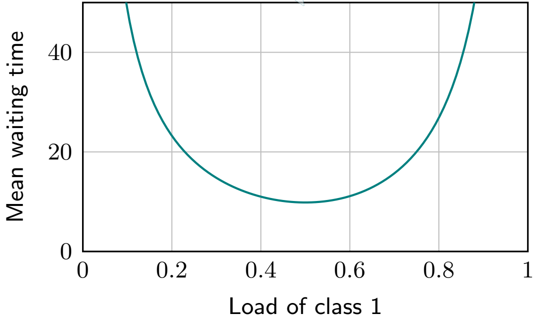

1
2
3
4
5
6
7
8
9

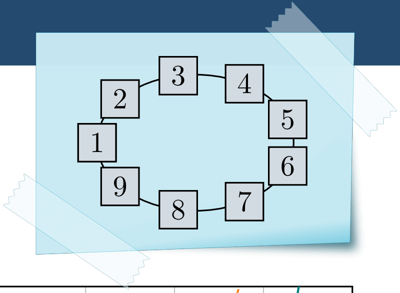

40
20
0
Mean waiting time
0
0.2
0.4
0.6
0.8
1
Load of class 1

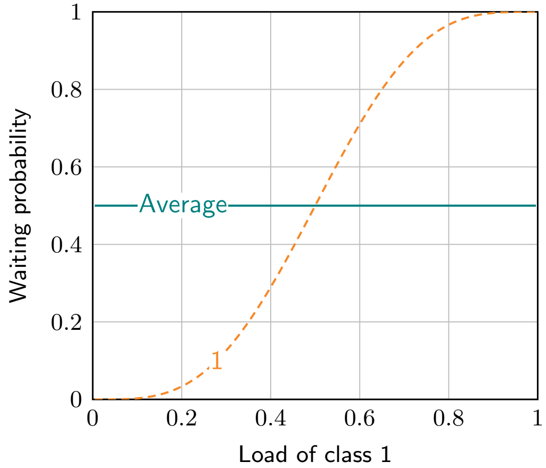
Numerical results: Cycle

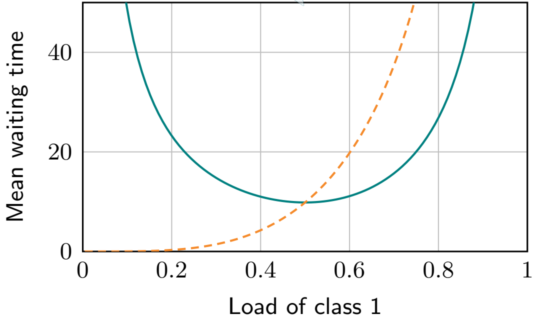




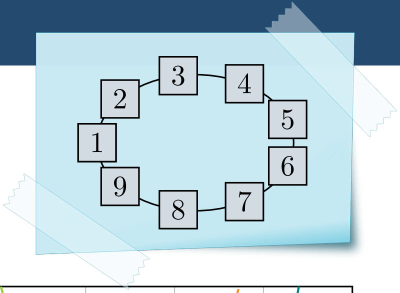


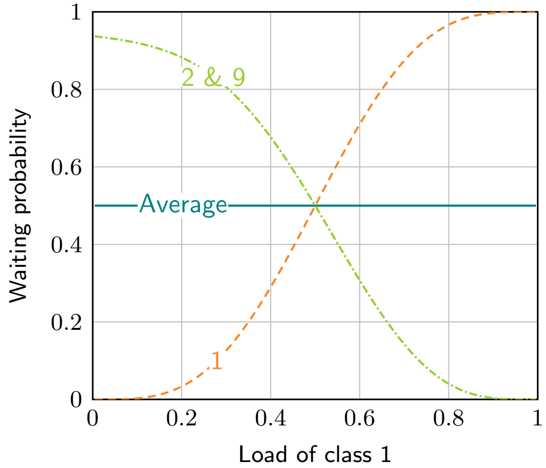

1
2
3
4
5
6
7
8
9

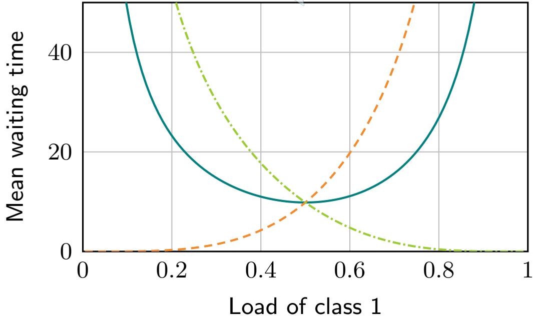

Waiting probability
1
0.8
0.6
0.4
0.2
0
Average
1
0
0.2
0.4
0.6
0.8
1
Load of class 1


Mean waiting time
40
20
0
0
0.2
0.4
0.6
0.8
1
Load of class 1

Numerical results: Cycle







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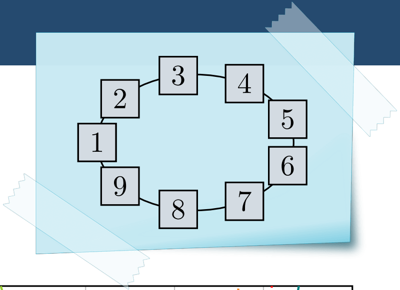
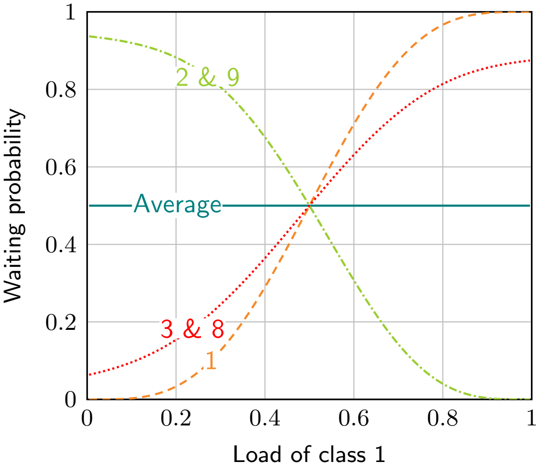
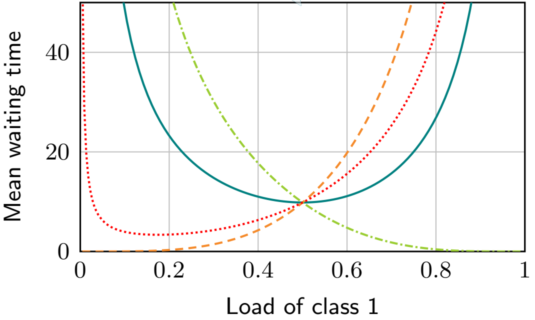
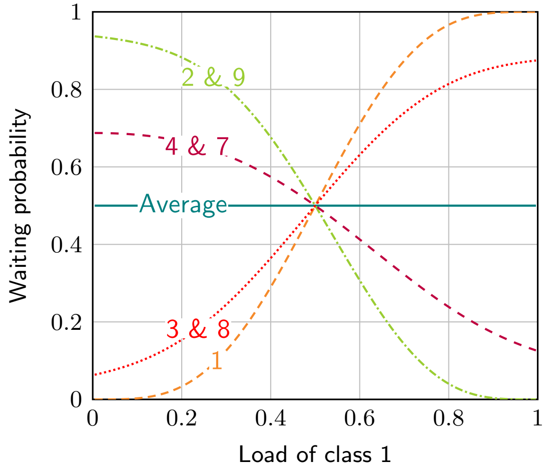
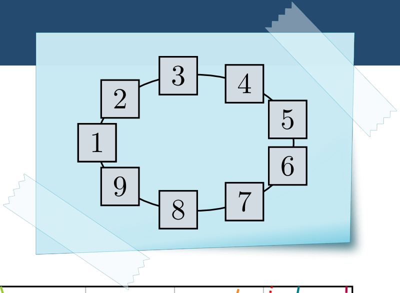


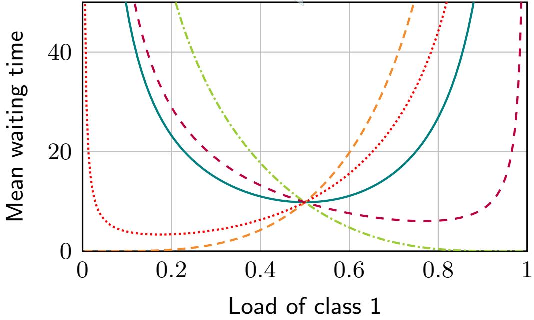


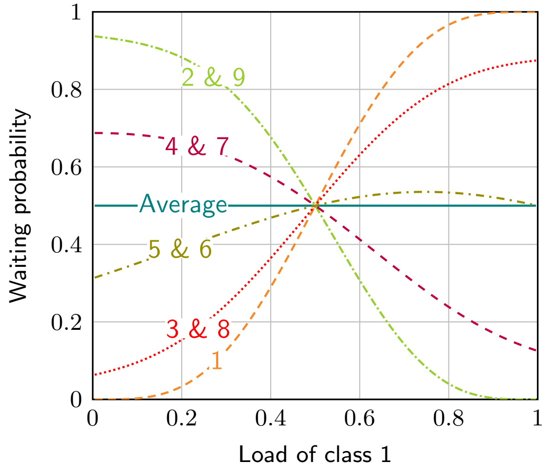


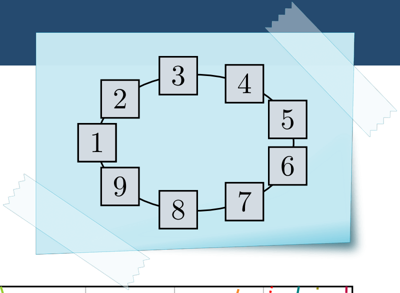
Numerical results: Cycle

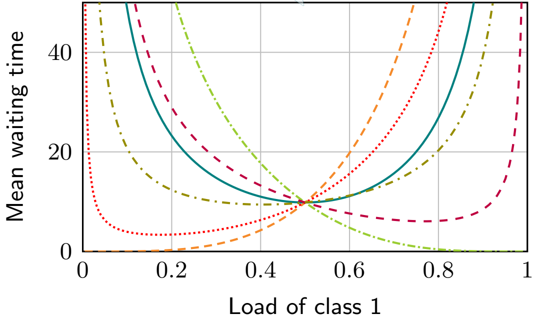




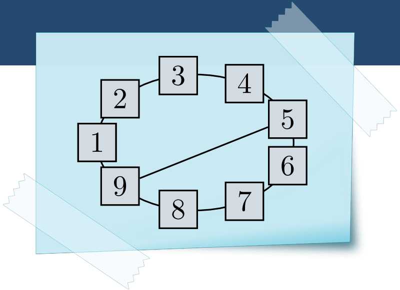



Waiting probability
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
Load of class 1
2 & 9
4 & 7
Average
5 & 6
3 & 8
1


1
2
3
4
5
6
7
8
9


Mean waiting time
40
20
0
0
0.2
0.4
0.6
0.8
1
Load of class 1

Numerical results: Cycle with a chord



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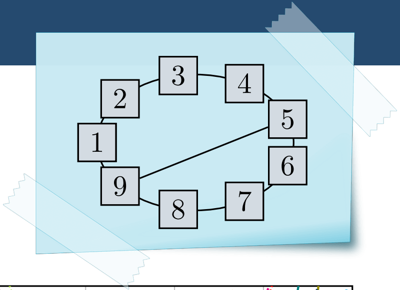
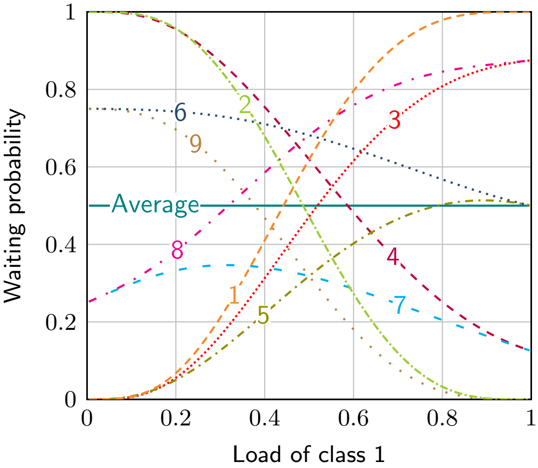
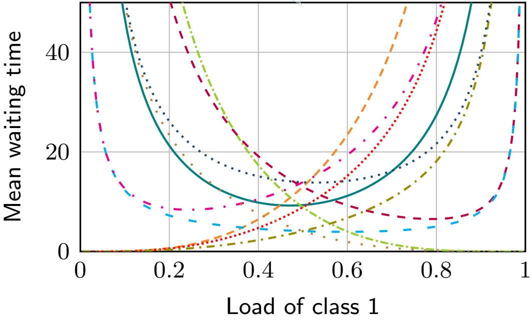




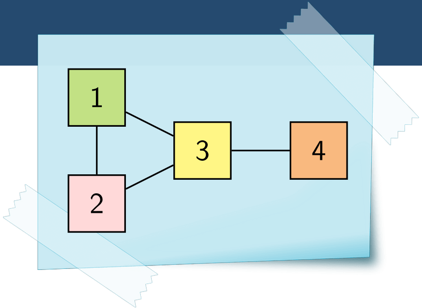


Outline

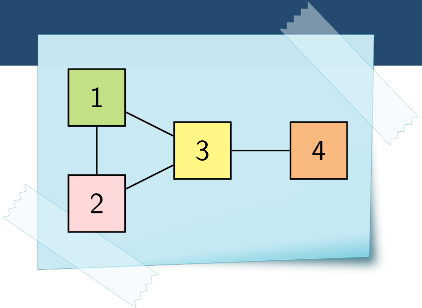
1. **Stochastic Matching: model, motivation, and notation**
2. **Performance under the first-come-first-matched policy**
   Comte, Stochastic Models (2022)
3. **Matching rates under an arbitrary policy**
   Comte, Mathieu, and Bušić, arXiv:2112.14457 (2022)

Matching rates

- **Matching rate** λ_k along edge $k = \{i, j\}$: mean number of matches per time unit between classes i and j.

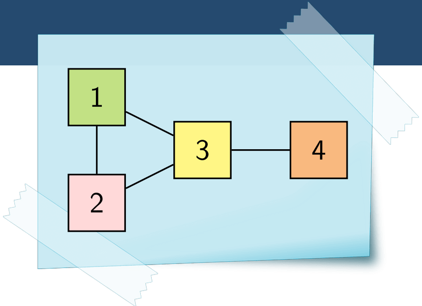


Matching rates



- **Matching rate** λ_k along edge $k = \{i, j\}$: mean number of matches per time unit between classes i and j.
- Matching rates are particularly interesting:
  - We often want to optimize a function of these matching rates.
  - They give intuition about the long-term impact of the matching policy.

Matching rates



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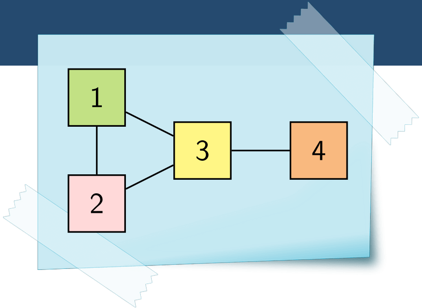
- Matching rates are particularly interesting:
  - We often want to optimize a function of these matching rates.
  - They give intuition about the long-term impact of the matching policy.

Given a graph $G = (V, E)$ and a vector $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ of arrival rates, what is the set of "feasible" vectors $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of matching rates?

Conservation equation



The matching rates satisfy the **conservation law**

$$\sum_{k \in E_i} \lambda_k = \mu_i, \quad i \in \{1, 2, \ldots, n\}.$$

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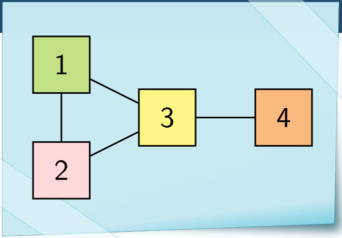
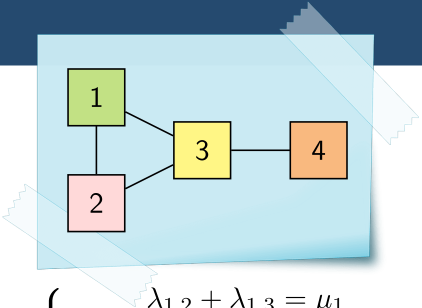


$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{3,4} = \mu_4 \end{cases}$$

Conservation equation



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that is, in matrix form,

$$A\lambda = \mu,$$

where $A = (a_{i,k})$ is the incidence matrix of the compatibility graph.

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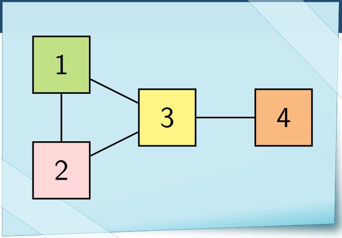

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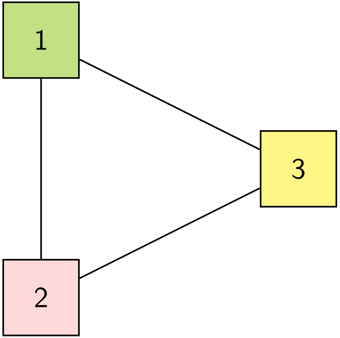
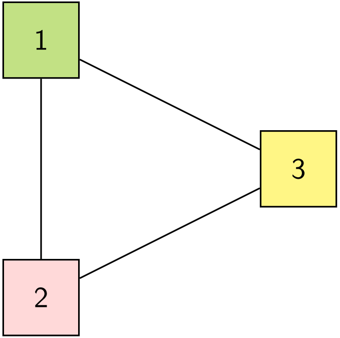
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

Example: Triangle graph



$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} = \mu_3 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$



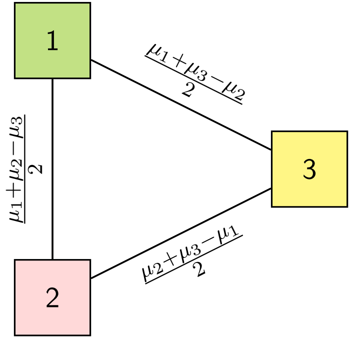
$$\lambda_{1,2} = \frac{\mu_1 + \mu_2 - \mu_3}{2}$$
$$\lambda_{1,3} = \frac{\mu_1 + \mu_3 - \mu_2}{2}$$
$$\lambda_{2,3} = \frac{\mu_2 + \mu_2 - \mu_1}{2}$$

$$\leftarrow$$

$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} = \mu_3 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

Example: Triangle graph

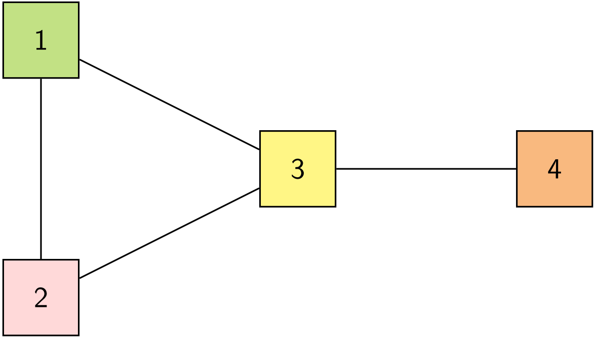


$$\leftarrow \quad \lambda_{1,2} = \frac{\mu_1 + \mu_2 - \mu_3}{2}, \quad \lambda_{1,3} = \frac{\mu_1 + \mu_3 - \mu_2}{2}, \quad \lambda_{2,3} = \frac{\mu_2 + \mu_2 - \mu_1}{2} \quad \leftarrow$$

$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} = \mu_3 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

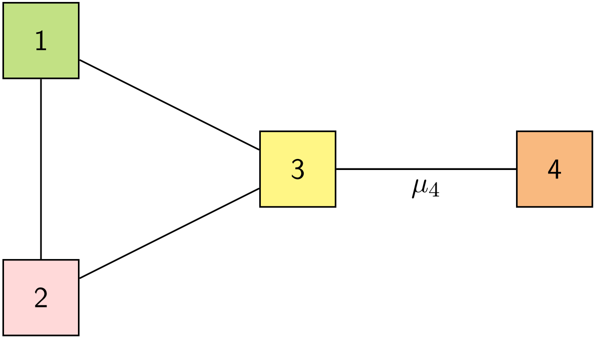
Example: Paw graph



$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{3,4} = \mu_4 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

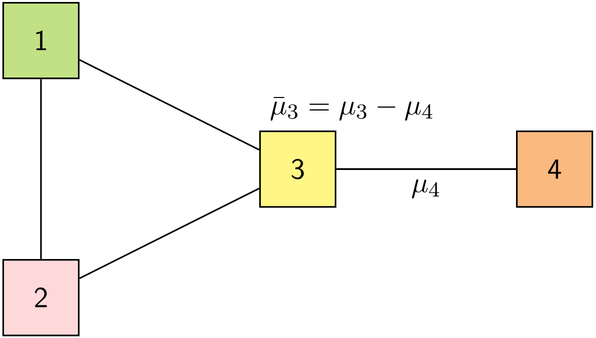
Example: Paw graph



$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{3,4} = \mu_4 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

Example: Paw graph



$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{3,4} = \mu_4 \end{cases}$$

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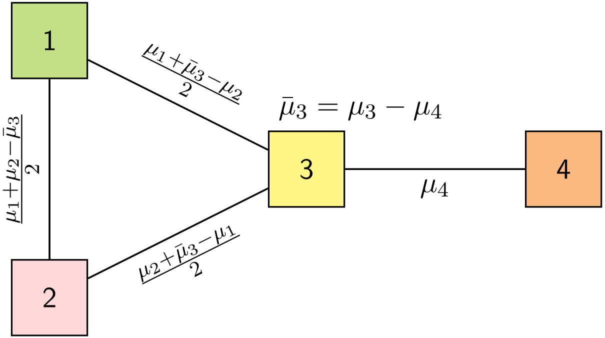
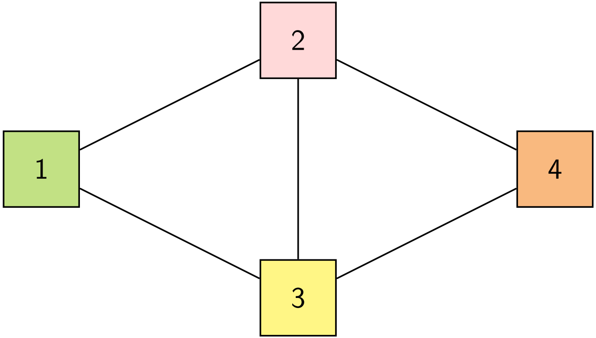
Example: Paw graph



$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{3,4} = \mu_4 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

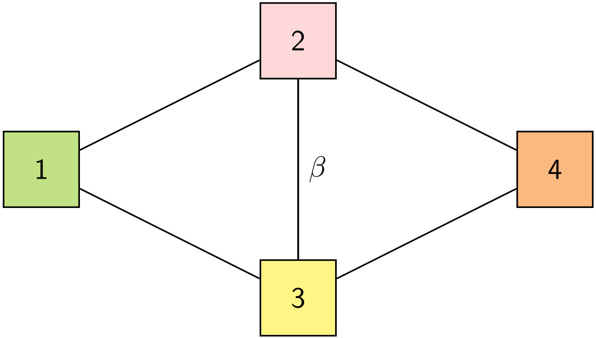
Example: Diamond graph



$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} + \lambda_{2,4} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{2,4} + \lambda_{3,4} = \mu_4 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

Example: Diamond graph

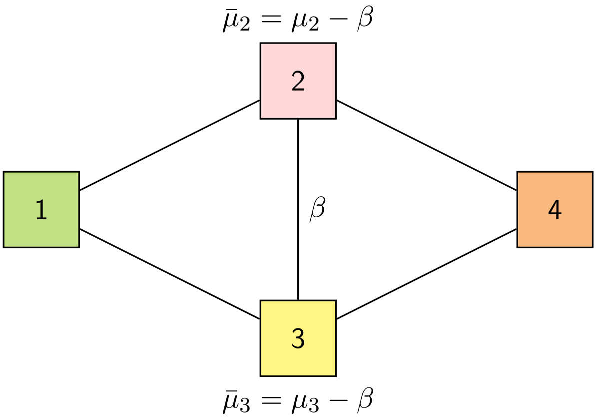


$$\beta = \frac{1}{2}(\mu_2 + \mu_3 - \mu_1 - \mu_4)$$

$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} + \lambda_{2,4} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{2,4} + \lambda_{3,4} = \mu_4 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

Example: Diamond graph

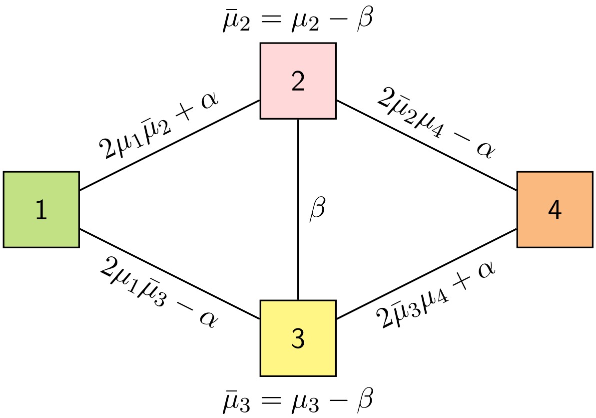


$$\beta = \tfrac{1}{2}(\mu_2 + \mu_3 - \mu_1 - \mu_4)$$
$$\mu_1 + \mu_4 = \bar\mu_2 + \bar\mu_3 = \tfrac{1}{2}$$

$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} + \lambda_{2,4} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{2,4} + \lambda_{3,4} = \mu_4 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

Example: Diamond graph



$$\beta = \frac{1}{2}(\mu_2 + \mu_3 - \mu_1 - \mu_4)$$
$$\mu_1 + \mu_4 = \bar\mu_2 + \bar\mu_3 = \frac{1}{2}$$

$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} + \lambda_{2,4} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{2,4} + \lambda_{3,4} = \mu_4 \end{cases}$$

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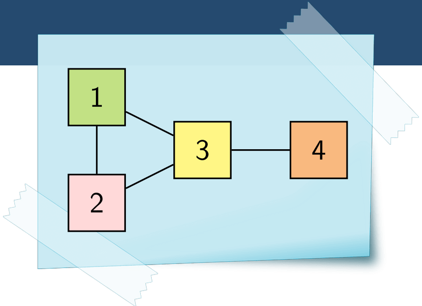
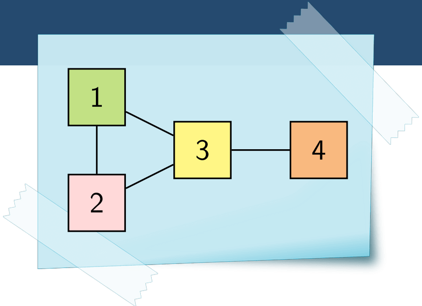
Surjectivity, injectivity, and bijectivity



- The compatibility graph G is **surjective** if
  - The linear application $\lambda \in \mathbb{R}^m \mapsto A\lambda \in \mathbb{R}^n$ is surjective.

Surjectivity, injectivity, and bijectivity



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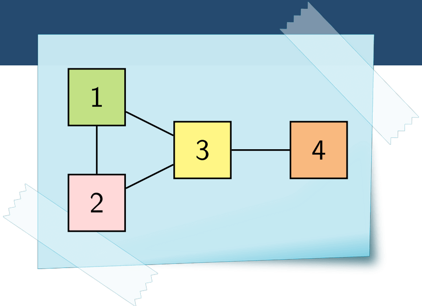
- The compatibility graph G is **injective** if
  - The linear application $\lambda \in \mathbb{R}^m \mapsto A\lambda \in \mathbb{R}^n$ is injective.

Surjectivity, injectivity, and bijectivity



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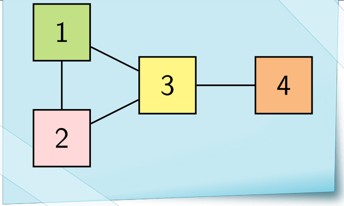
- The compatibility graph G is **injective** if
  - The linear application $\lambda \in \mathbb{R}^m \mapsto A\lambda \in \mathbb{R}^n$ is injective.

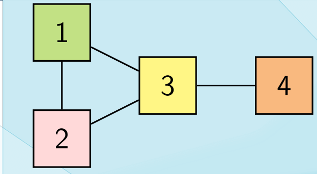
- The compatibility graph G is **bijective** if G is surjective and injective.

Surjectivity, injectivity, and bijectivity



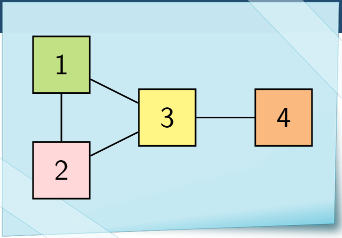
- The compatibility graph G is **surjective** if
  - The linear application $\lambda \in \mathbb{R}^m \mapsto A\lambda \in \mathbb{R}^n$ is surjective.
  - The conservation law $A\lambda = \mu$ has at least one solution, for each $\mu \in \mathbb{R}^n$.
- The compatibility graph G is **injective** if
  - The linear application $\lambda \in \mathbb{R}^m \mapsto A\lambda \in \mathbb{R}^n$ is injective.
  - The conservation law $A\lambda = \mu$ has at most one solution, for each $\mu \in \mathbb{R}^n$.
- The compatibility graph G is **bijective** if G is surjective and injective.

Surjectivity, injectivity, and bijectivity



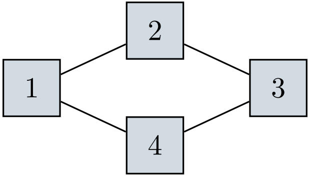
- The compatibility graph G is **surjective** if
  - The linear application $\lambda \in \mathbb{R}^m \mapsto A\lambda \in \mathbb{R}^n$ is surjective.
  - The conservation law $A\lambda = \mu$ has at least one solution, for each $\mu \in \mathbb{R}^n$.
  - **The compatibility graph G is non-bipartite** (i.e., contains at least one odd cycle).
- The compatibility graph G is **injective** if
  - The linear application $\lambda \in \mathbb{R}^m \mapsto A\lambda \in \mathbb{R}^n$ is injective.
  - The conservation law $A\lambda = \mu$ has at most one solution, for each $\mu \in \mathbb{R}^n$.
- The compatibility graph G is **bijective** if G is surjective and injective.

Surjectivity, injectivity, and bijectivity

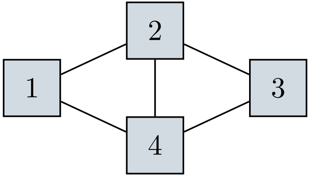


- The compatibility graph G is **surjective** if
  - The linear application $\lambda \in \mathbb{R}^m \mapsto A\lambda \in \mathbb{R}^n$ is surjective.
  - The conservation law $A\lambda = \mu$ has at least one solution, for each $\mu \in \mathbb{R}^n$.
  - **The compatibility graph G is non-bipartite** (i.e., contains at least one odd cycle).
- The compatibility graph G is **injective** if
  - The linear application $\lambda \in \mathbb{R}^m \mapsto A\lambda \in \mathbb{R}^n$ is injective.
  - The conservation law $A\lambda = \mu$ has at most one solution, for each $\mu \in \mathbb{R}^n$.
  - **The compatibility graph G contains at most one cycle and this cycle is odd**.
- The compatibility graph G is **bijective** if G is surjective and injective.

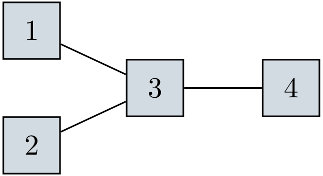
Surjectivity, injectivity, and bijectivity



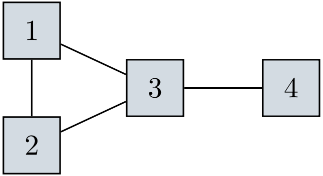
(a) Neither surjective, nor injective



(b) Surjective-only

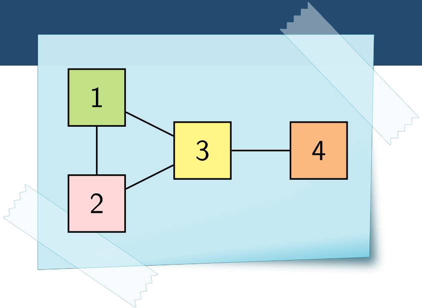


(c) Injective only



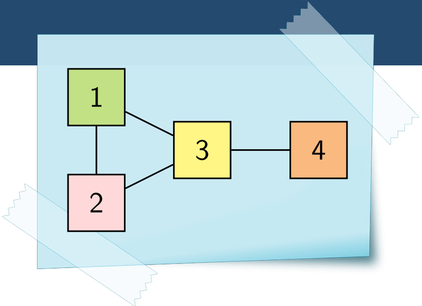
(d) Bijective

"Stabilizability"



$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

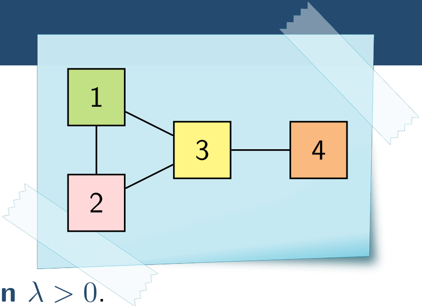
"Stabilizability"



- A matching problem (G, μ) is **stabilizable** if and only if $\rho(I) < 1$ for each $I \in \mathbb{I}$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

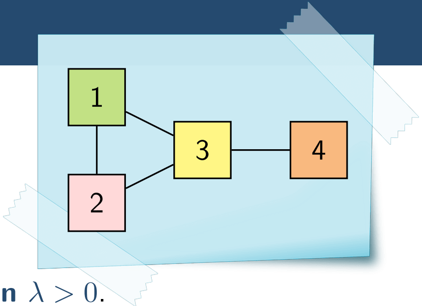
"Stabilizability"



- A matching problem (G, μ) is **stabilizable** if and only if **the conservation law** $A\lambda = \mu$ **has a solution** $\lambda > 0$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

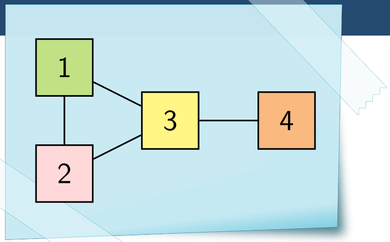
"Stabilizability"



- A matching problem (G, μ) is **stabilizable** if and only if **the conservation law** $A\lambda = \mu$ **has a solution** $\lambda > 0$.
  ☺ The time complexity to verify this condition is polynomial in n and m.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

“Stabilizability”

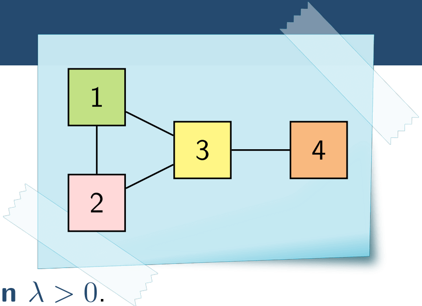


- A matching problem (G, μ) is **stabilizable** if and only if **the conservation law $A\lambda = \mu$ has a solution $\lambda > 0$.**
  ☺ The time complexity to verify this condition is polynomial in n and m.

- A compatibility graph G is **stabilizable** if and only if G is non-bipartite.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

"Stabilizability"

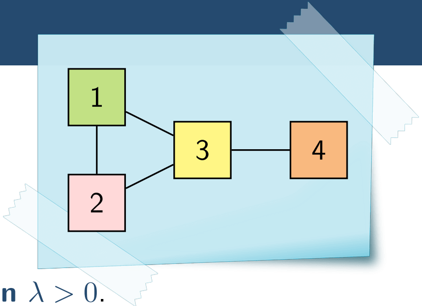


- A matching problem (G, μ) is **stabilizable** if and only if **the conservation law** $A\lambda = \mu$ **has a solution** $\lambda > 0$.
  ☺ The time complexity to verify this condition is polynomial in n and m.

- A compatibility graph G is **stabilizable** if and only if G **is surjective**.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

"Stabilizability"

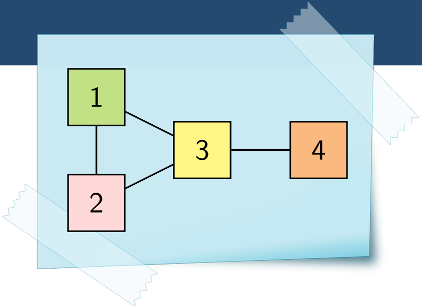


- A matching problem (G, μ) is **stabilizable** if and only if **the conservation law $A\lambda = \mu$ has a solution** $\lambda > 0$.
  ☺ The time complexity to verify this condition is polynomial in n and m.

- A compatibility graph G is **stabilizable** if and only if G **is surjective**.
  ☺ The rank of matrix A is n.
  The nullity of matrix A is $d = m - n$
  (according to the rank-nullity theorem).

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

Affine space of solutions

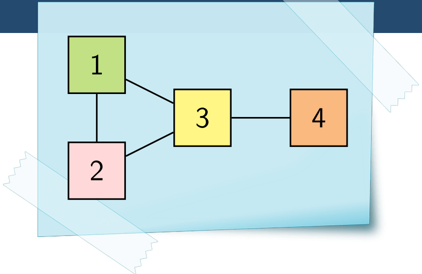


- The solution set of the conservation law $A\lambda = \mu$ is

$$\Lambda = \left\{ \lambda^\circ + \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_d b_d : \alpha \in \mathbb{R}^d \right\}$$

where λ° is a particular solution of the conservation law and $\{b_1, b_2, \ldots, b_d\}$ is a basis of $\mathrm{Ker}(A)$, of cardinality $d = m - n$.

Affine space of solutions



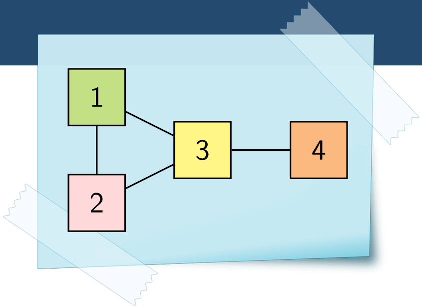
- The solution set of the conservation law $A\lambda = \mu$ is

$$\Lambda = \left\{ \lambda^\circ + \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_d b_d : \alpha \in \mathbb{R}^d \right\}$$

  where λ° is a particular solution of the conservation law and $\{b_1, b_2, \ldots, b_d\}$ is a basis of $\mathrm{Ker}(A)$, of cardinality $d = m - n$.

- We borrowed an algorithm from (Doob, 1973) to build a basis of $\mathrm{Ker}(A)$.

Affine space of solutions

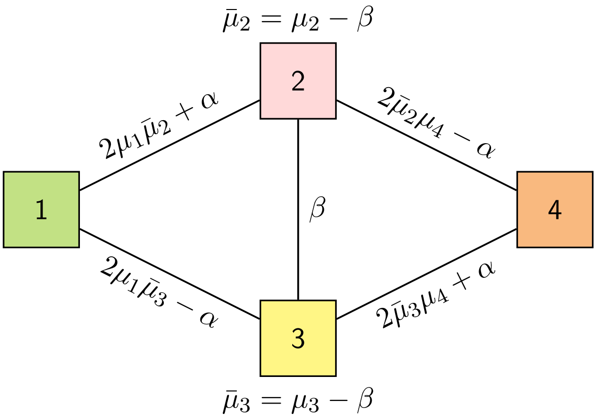


- The solution set of the conservation law $A\lambda = \mu$ is

$$\Lambda = \left\{ \lambda^\circ + \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_d b_d : \alpha \in \mathbb{R}^d \right\}$$

  where λ° is a particular solution of the conservation law and $\{b_1, b_2, \ldots, b_d\}$ is a basis of $\mathrm{Ker}(A)$, of cardinality $d = m - n$.
- We borrowed an algorithm from (Doob, 1973) to build a basis of $\mathrm{Ker}(A)$.
- We use two coordinate systems:
  - **Edge** coordinates $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m$.
  - **Kernel** coordinates $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$.

Example: Diamond graph

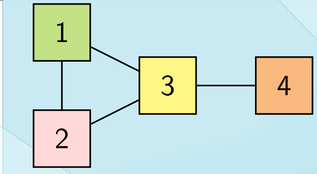


$$\beta = \tfrac{1}{2}(\mu_2 + \mu_3 - \mu_1 - \mu_4)$$
$$\mu_1 + \mu_4 = \bar{\mu}_2 + \bar{\mu}_3 = \tfrac{1}{2}$$

$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} + \lambda_{2,4} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{2,4} + \lambda_{3,4} = \mu_4 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

Polytope of non-negative solutions

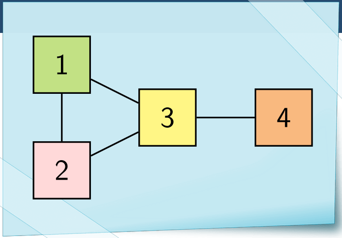


- The set of non-negative solutions of the conservation law is

$$\Lambda_{\geq 0} = \Lambda \cap \mathbb{R}_+^m$$
$$\approx \left\{ \alpha \in \mathbb{R}^d : \lambda^\circ + \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_d b_d \geq 0 \right\}.$$

  This is a **d-dimensional convex polytope**.

Polytope of non-negative solutions



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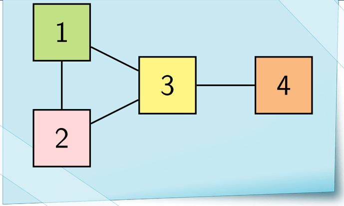
- The subgraph restricted to the support of a vertex of $\Lambda_{\geq 0}$ is injective

Polytope of non-negative solutions



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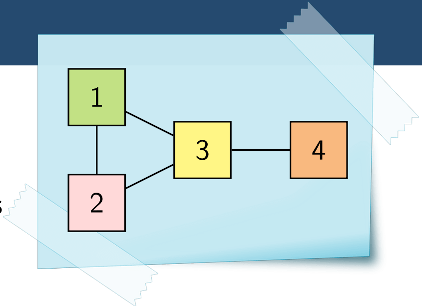
Polytope of non-negative solutions



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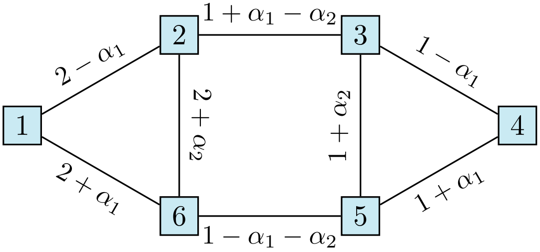
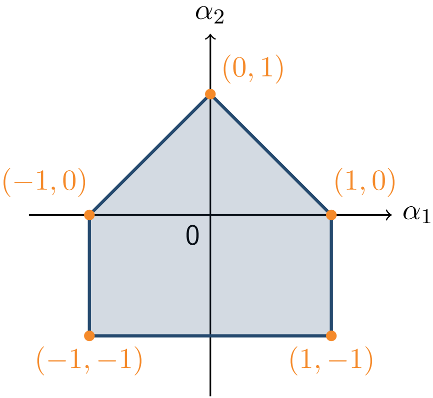
- The subgraph restricted to the support of a vertex of $\Lambda_{\geq 0}$ is injective:
  - If the subgraph is bijective, the vertex is achieved by any stable policy applied to the subgraph.
  - If the subgraph is injective but not surjective, it's more complicated...

Example: Codomino graph

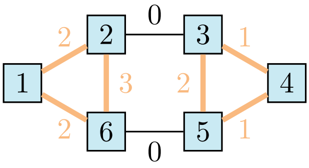


(a) Solution of the conservation law $A\lambda = \mu$.

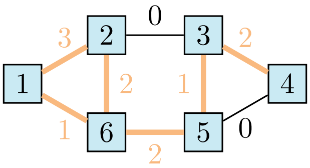


(b) Polytope $\Lambda_{\geq 0}$ in kernel coordinates.

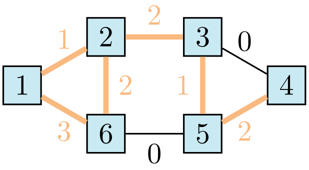
Example: Codomino graph



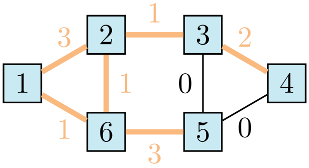
(a) Vertex $(0, 1)$.



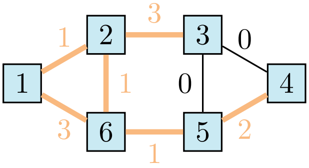
(b) Vertex $(-1, 0)$.



(c) Vertex $(1, 0)$.

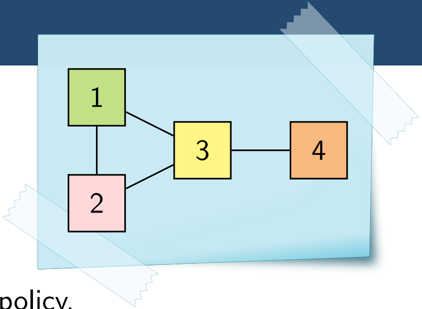


(d) Vertex $(-1, -1)$.



(e) Vertex $(1, -1)$.

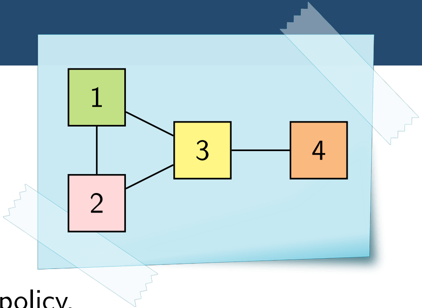
Conclusion



Take-away

- Stochastic dynamic matching problem associated with organ transplant programs and assembly systems.
- Performance evaluation under the first-come-first-matched policy.
- Analysis of the matching rates under an arbitrary matching policy.

Conclusion



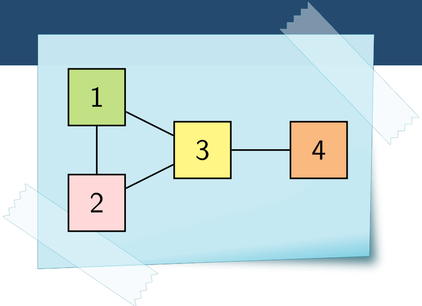
Take-away

- Stochastic dynamic matching problem associated with organ transplant programs and assembly systems.
- Performance evaluation under the first-come-first-matched policy.
- Analysis of the matching rates under an arbitrary matching policy.

Future works

- More realistic model: hypergraph? state-dependent arrival rates?
- Optimization and learning: graph structure? arrival rates? policy?

References



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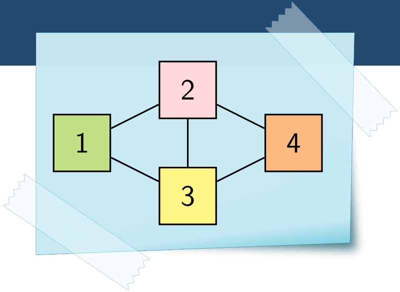
C. Comte and J.-P. Dorsman. "Performance Evaluation of Stochastic Bipartite Matching Models". *Performance Engineering and Stochastic Modeling*. Lecture Notes in Computer Science. Springer, 2021, pp. 425–440

C. Comte, F. Mathieu, and A. Bušić. "Stochastic dynamic matching: A mixed graph-theory and linear-algebra approach". (Jan. 2022). arXiv: 2112.14457

Basis of the kernel of the matrix A



- A vector $\lambda \in \mathbb{R}^m$ belongs to $\mathrm{Ker}(A)$ if and only if $A\lambda = 0$.

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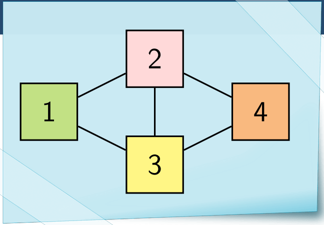

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$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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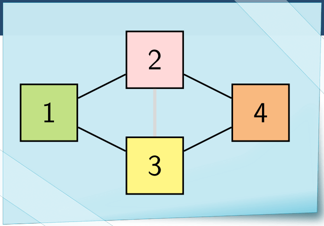

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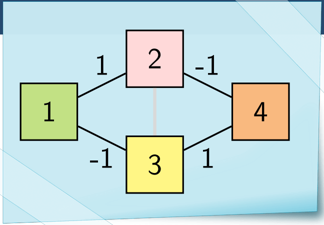

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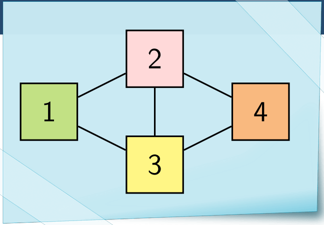

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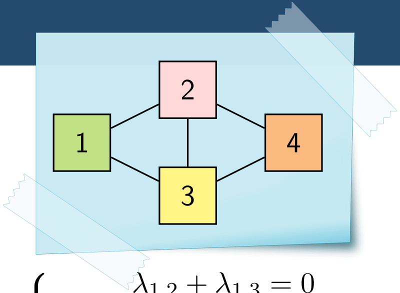
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Basis of the kernel of the matrix A

- A vector $\lambda \in \mathbb{R}^m$ belongs to Ker(A) if and only if $A\lambda = 0$.
- Algorithm to construct a basis of Ker(A) (Doob, 1973)



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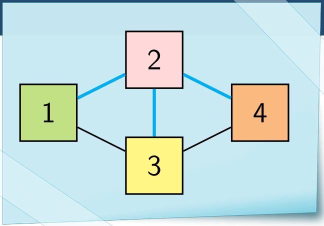

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  1. Build a spanning tree $\mathbf{T}$ of G.

$$
\begin{cases}
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\lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = 0 \\
\lambda_{2,4} + \lambda_{3,4} = 0
\end{cases}
$$

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
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\end{bmatrix}
\begin{bmatrix}
\lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

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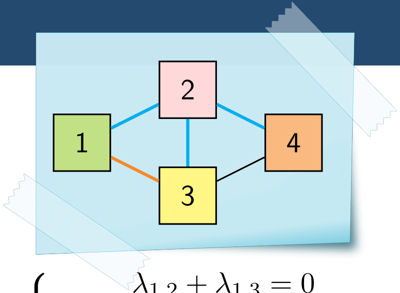

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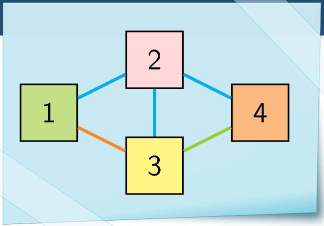

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  3. For each edge $\mathbf{l} \notin (\mathbf{T} \cup \{\mathbf{k}\})$, build a kernel vector with support $\{\mathbf{l}\} \subseteq S \subseteq \mathbf{T} \cup \{\mathbf{k}, \mathbf{l}\}$

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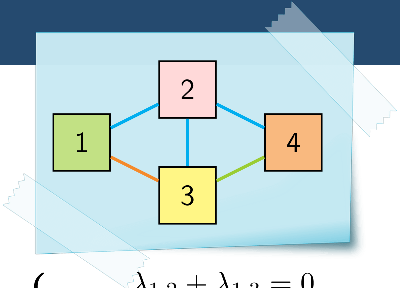

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  3. For each edge $\mathbf{l} \notin (\mathbf{T} \cup \{\mathbf{k}\})$, build a kernel vector with support $\{\mathbf{l}\} \subseteq S \subseteq \mathbf{T} \cup \{\mathbf{k}, \mathbf{l}\}$
- The matching rate along an edge is unique if and only if this edge doesn't belong to any "generalized even cycle".

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